Maxima and Minima

1. Let $\$ y$ be the total cost of transportation .

The train travels from C along side railway CA and main railway AB.

$\angle CAM = \theta$, CM = a, BM = b .

$$y=\left(\frac{a}{\sin\theta}\right)p+\left[b-\frac{a}{\tan\theta}\right]q$$

$$\frac{dy}{d\theta}=-\frac{ap\cos\theta}{\sin^2\theta}+\frac{aq\sec^2\theta}{\tan^2\theta}=\frac{-ap\cos\theta+aq}{\sin^2\theta}=-a\frac{p\cos\theta-q}{\sin^2\theta}$$

When $\cos\theta=\frac{q}{p}$, $\frac{dy}{d\theta}=0$, where $p > q$

When $0<\theta<\cos^{-1}\frac{q}{p}$, $\cos\theta>\frac{p}{q}$, $p\cos\theta-q>0$, $\frac{dy}{d\theta}<0$. (Note that $\cos\theta$ is decreasing)

When $\frac{\pi}{2}>\theta>\cos^{-1}\frac{q}{p}$, $\cos\theta<\frac{p}{q}$, $p\cos\theta-q<0$, $\frac{dy}{d\theta}>0$.

$\therefore$ y is a min. when $\theta=\cos^{-1}\frac{q}{p}$.

An interpretation in the theory of waves : critical angle in Snell's Law.

2. **Method 1** $4x^2 + 6xy + 9y^2 - 8x - 24y + 4 = 0$ …. (1)

$\Rightarrow$ $8x+6x\frac{dy}{dx}+6y+18y\frac{dy}{dx}-8-24\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=\frac{8-8x-6y}{6x+18y-24}=\frac{4-4x-3y}{3x+6y-8}$ .... (2)

$\frac{dy}{dx}=0$ $\Rightarrow$ $4 - 4x - 3y = 0$ $\Rightarrow$ $y=\frac{4-4x}{3}$ .... (3)

(3)↓(1), $4x^2+6x\left(\frac{4-4x}{3}\right)+9\left(\frac{4-4x}{3}\right)^2-8x-24\left(\frac{4-4x}{3}\right)+4=0$ $\Rightarrow x = 1$ or -1 …. (4)

When x = 1, y = 0 , $\frac{dy}{dx}=0$ When x = -1 , y = 8/3 , $\frac{dy}{dx}=0$.

From (2), $\frac{d^2y}{dx^2}=\frac{(3x+6y-8)\left(-4-3\frac{dy}{dx}\right)-(4-4x-3y)\left(3+6\frac{dy}{dx}\right)}{(3x+6y-8)^2}$

$\left.\frac{d^2y}{dx^2}\right|_{(1,0)}=\frac{4}{5}>0\Rightarrow$ $y_{min} = 0$. $\left.\frac{d^2y}{dx^2}\right|_{\left(-1,\frac{8}{3}\right)}=-\frac{4}{5}<0\Rightarrow$ $y_{max} = 8/3$.

Method 2 $4x^2 + 6xy + 9y^2 - 8x - 24y + 4 = 0$ $\Rightarrow$ $4x^2 + (6y - 8)x + (9y^2 - 24y + 4) = 0$

x is real $\Rightarrow$ $\Delta = (6y - 8)^2 - 4(4)(9y^2 - 24y + 4) \ge 0$ $\Rightarrow$ $108y^2 - 288y \le 0$ $\Rightarrow$ $y(3y - 8) \le 0$

$\Rightarrow$ $0\le y\le\frac{8}{3}$ $\Rightarrow$ $y_{min} = 0$ and $y_{max} = 8/3$.

4. $y = k\left[\frac{x}{\sqrt{x^2+a^2}}-\frac{x}{\sqrt{x^2+b^2}}\right]$, where k is a constant .

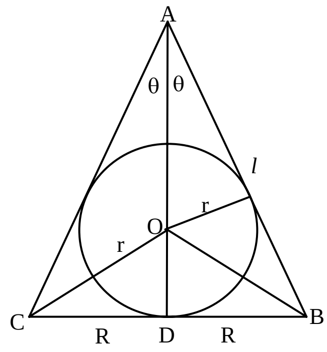
$$\frac{dy}{dx}=k\left[\frac{a^2}{(x^2+a^2)^{3/2}}-\frac{b^2}{(x^2+b^2)^{3/2}}\right]=0\Rightarrow\frac{a^2}{(x^2+a^2)^{3/2}}=\frac{b^2}{(x^2+b^2)^{3/2}}\Rightarrow a^2(x^2+b^2)^{3/2}=b^2(x^2+a^2)^{3/2}$$

$$\Rightarrow a^{4/3}(x^2+b^2)=b^{4/3}(x^2+a^2)\Rightarrow x^2=\frac{a^2b^{4/3}-a^{4/3}b^2}{a^{4/3}-b^{4/3}}\Rightarrow x=\sqrt{\frac{a^2b^{4/3}-a^{4/3}b^2}{a^{4/3}-b^{4/3}}}\quad, x>0 .$$

By replacing the above with inequalities, we can check that $\frac{dy}{dx}$ changes from positive to negative.

Therefore y is a max. at the above value of x.

5. The cross section of conical shell is shown on the right.



$$OA = \frac{r}{\sin\theta}, \qquad AD = OA + OD = \frac{r}{\sin\theta} + r = \frac{r(1+\sin\theta)}{\sin\theta}$$

$$l = \frac{AD}{\cos\theta} = \frac{r(1+\sin\theta)}{\sin\theta\cos\theta}, \qquad R = AD\tan\theta = \frac{r(1+\sin\theta)}{\sin\theta}\tan\theta = \frac{r(1+\sin\theta)}{\cos\theta}$$

Area of conical shell = A = $\pi R l$

$$= \pi\frac{r(1+\sin\theta)}{\sin\theta\cos\theta}\frac{r(1+\sin\theta)}{\cos\theta} = \pi r^2\frac{(1+\sin\theta)^2}{\sin\theta\cos^2\theta} = \pi r^2\frac{1+\sin\theta}{\sin\theta(1-\sin\theta)}$$

$$\frac{dA}{d\theta} = \pi r^2\frac{\sin\theta(1-\sin\theta)\cos\theta - (1+\sin\theta)[\cos\theta(1-\sin\theta)+\sin\theta(-\cos\theta)]}{\sin^2\theta(1-\sin\theta)^2}$$

$$= \pi r^2\cos\theta\frac{\sin\theta(1-\sin\theta)-(1+\sin\theta)[(1-\sin\theta)-\sin\theta]}{\sin^2\theta(1-\sin\theta)^2} = \pi r^2\cos\theta\frac{\sin^2\theta+2\sin\theta-1}{\sin^2\theta(1-\sin\theta)^2} = 0$$

$$\Rightarrow \quad \sin^2\theta + 2\sin\theta - 1 = 0 \qquad \Rightarrow \sin\theta = \frac{-2\pm\sqrt{8}}{2} = \sqrt{2}-1, \qquad \theta\in\left(0,\frac{\pi}{2}\right)$$

$\theta = \sin^{-1}\left(-1+\sqrt{2}\right) \approx 0.4271 \approx 24.47^{o}$ (Test for minimum omitted here)

$\therefore$ Area of conical shell = A

$$= \pi r^2\frac{1+\sin\theta}{\sin\theta(1-\sin\theta)} = \pi r^2\frac{1+\sqrt{2}-1}{(\sqrt{2}-1)(1-(\sqrt{2}-1))} = (3+2\sqrt{2})\pi r^2$$

6. Let h be the height and V be the volume of the cylindrical hole .

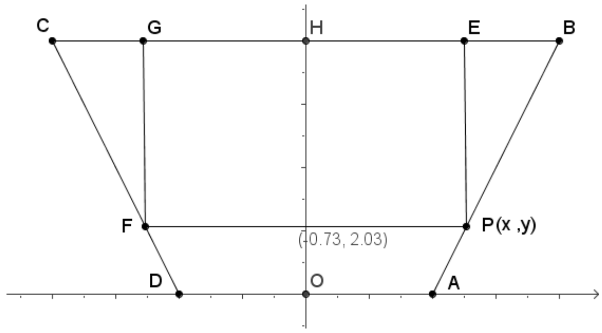
$$V = \pi(R^2-h^2)h = \pi R^2 h - \pi h^3 \qquad \Rightarrow \frac{dV}{dh} = \pi R^2 - 3\pi h^2 = \pi(R^2-3h^2)$$

$$\frac{dV}{dh} = 0 \Rightarrow h = \frac{R}{\sqrt{3}}, \qquad h < \frac{R}{\sqrt{3}} \Rightarrow \frac{dV}{dh} > 0, \qquad h > \frac{R}{\sqrt{3}} \Rightarrow \frac{dV}{dh} < 0$$

V = 0 when h = 0 or h = R. $\therefore$ V is a absolute max. when $h = \frac{R}{\sqrt{3}}$.

Metal remaining in this case = $\frac{1}{2}\left(\frac{4}{3}\pi R^3\right) - \pi\left(R^2 - \frac{R^2}{3}\right)\frac{R}{\sqrt{3}} = \frac{2(\sqrt{3}-1)}{3\sqrt{3}}\pi R^3$

7. The cross –section is shown on the right.



AD = a, BC = b, OH = L

PEGF is the cross-section of the cylinder.

$$A = \left(\frac{a}{2}, 0\right), \qquad B = \left(\frac{b}{2}, 0\right)$$

P(x, y) is a point on AB.

$y = \frac{2L}{b-a}\left(x - \frac{a}{2}\right)$ …. (1)

$V = \pi x^2(L-y) = \pi x^2\left[L - \frac{2L}{b-a}\left(x-\frac{a}{2}\right)\right] = \frac{\pi L}{b-a}\left[x^2(b-a) - 2x^3 + ax^2\right]$

$\frac{dV}{dx} = \frac{\pi L}{b-a}\left[2x(b-a) - 6x^2 + 2ax\right] = \frac{2\pi L}{b-a}x(b-3x)$

Obviously $x = 0$ is not the solution we want.

When $x = b/3$, $\frac{dV}{dx} = 0$. When $\frac{a}{2} < x < \frac{b}{3}, \frac{dV}{dx} > 0$. When $\frac{b}{3} < x < \frac{b}{2}, \frac{dV}{dx} < 0$.

V is a local max. when $x = b/3$

$V_{max} = \pi\left(\frac{b}{3}\right)^2\left[L - \frac{2L}{b-a}\left(\frac{b}{3} - \frac{a}{2}\right)\right] = \frac{\pi b^3 L}{27(b-a)}$.

When $x = a/2$, $V = \frac{\pi a^2 L}{4}$. When $x = b/2$, $V = 0$.

Since $b > \frac{3a}{2}$ and $b - a > 0$, it is not difficult to check that $\frac{\pi b^3 L}{27(b-a)} > \frac{\pi a^2 L}{4}$ and $V_{max} = \frac{\pi b^3 L}{27(b-a)}$ is the absolute maximum.

8. $V = \pi r^2 h = 75$

Surface area of can = A = $2\pi r^2 + 2\pi r\left(h + \frac{1}{2}\right) = 2\pi r^2 + 2\pi r\left(\frac{75}{\pi r^2} + \frac{1}{2}\right) = 2\pi r^2 + \pi r + \frac{150}{r}$

$\frac{dA}{dr} = 4\pi r + \pi - \frac{150}{r^2} = \frac{4\pi r^3 + \pi r^2 - 150}{r^2}$

$\frac{d^2A}{dr^2} = 4\pi + \frac{300}{r^3} > 0$ $\therefore$ A is a minimum when $\frac{dA}{dr} = 0$, or $4\pi r^3 + \pi r^2 - 150 = 0$.

9. $f'(a) = 0$ and $f''(a) > 0$.

$\therefore$ $f''(a) = \lim_{\Delta x \to 0}\frac{f'(a+\Delta x) - f'(a)}{\Delta x} > 0$

When $\Delta x \to 0^+$, $f'(a+\Delta x) - f'(a) > 0 \Rightarrow f'(a+\Delta x) > f'(a) = 0$

When $\Delta x \to 0^-$, $f'(a+\Delta x) - f'(a) < 0 \Rightarrow f'(a+\Delta x) < f'(a) = 0$

$\therefore$ f(x) is decreasing before a locally and is increasing after a locally.

$\therefore$ f(x) has a minimum value for $x = a$.

Let $y = \cos 9\theta \sec^2\theta$

$y' = \cos 9\theta \sec\theta (\sec\theta \tan\theta) + \sec^2\theta(-9\sin 9\theta) = \sec^2\theta(2\tan\theta\cos 9\theta - 9\sin 9\theta)$

$y' = 0 \Rightarrow 2\tan\theta\cos 9\theta - 9\sin 9\theta = 0 \Rightarrow 2\tan\theta = 9\tan 9\theta$

$\Rightarrow \theta \approx 0.3583, 0.7196, 1.0922$ (θ between 0 and $\frac{1}{2}\pi$)

$y = \cos 9\theta \sec^2\theta$ is a minimum for $\theta \approx 0.3583$ and 1.0922

by checking the sign change of $2\tan\theta\cos 9\theta - 9\sin 9\theta$ before and after the values.

10. Let $f(x) = \tan x - \left(x + \frac{x^3}{3}\right)$ for $x \in \left(0, \frac{\pi}{2}\right)$

$f'(x) = \sec^2 x - 1 - x^2 = \tan^2 x - x^2 > 0$ since $\tan x > x$ for $x \in \left(0, \frac{\pi}{2}\right)$.

$\therefore$ f(x) is strictly increasing and $f(x) > f(0) = 0$. Therefore $x + \frac{x^3}{3} < \tan x$ for $x \in \left(0, \frac{\pi}{2}\right)$.

11. $f(\theta) = \left(1 - \frac{\theta^2}{2}\right)\cos\theta + \theta\sin\theta$

$f'(\theta) = \left(1 - \frac{\theta^2}{2}\right)(-\sin\theta) + (-\theta)\cos\theta + \theta\cos\theta + \sin\theta = \frac{\theta^2}{2}\sin\theta > 0 \quad \forall\theta \in (0, \pi)$

12. Let $f(x) = \tan^{-1} x - \left(x - \frac{x^3}{3}\right)$, $\quad g(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \tan^{-1} x$, $x \in \left(0, \frac{\pi}{2}\right)$

$f'(x) = \frac{1}{1+x^2} - 1 + x^2 = \frac{x^4}{1+x^2} > 0$, $\quad g'(x) = 1 - x^2 + x^4 - \frac{1}{1+x^2} = \frac{x^6}{1+x^2} > 0$

$\therefore$ f(x) and g(x) are increasing .

$\therefore$ $f(x) > f(0) = 0$ and $g(x) > g(0) = 0$. Result follows.

13. Let $g(x) = e^{-x} - 1 + x$, then $g'(x) = -e^{-x} + 1 = 0$ when $x = 0$.

Also, $g''(x) = e^{-x} \Rightarrow g'(0) = 1 > 0$ $\therefore$ g(x) is min. when $x = 0$.

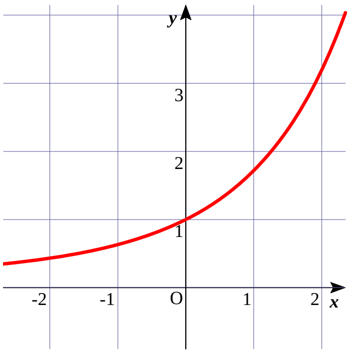
$g(x) = e^{-x} - 1 + x \geq g(0) = e^0 - 1 + 0 = 0$. and g(x) is never negative .

$f(x) = \frac{e^x - 1}{x} \Rightarrow f'(x) = \frac{xe^x - e^x + 1}{x^2} = \frac{e^x g(x)}{x^2} \geq 0$,

since $g(x) \geq 0$, $x^2 > 0$, $e^x > 0$ for all $x \neq 0$.

$\therefore$ f(x) is an increasing function of x for all x .

Graph of $y = f(x)$ is shown in R.H.S.



14. **(i)** Let $f(x) = \tan x - x$. Then $f'(x) = \sec^2 x - 1 = \tan^2 x > 0$, for all $0 < x < \frac{\pi}{2}$.

$\therefore$ f(x) is a strictly increasing function.

$\therefore$ $f(x) = \tan x - x > f(0) = \tan 0 - 0 = 0$. Result follows.

(ii) Let $f(x) = \frac{x}{\sin x}$. Then $f'(x) = \frac{\sin x - x\cos x}{\sin^2 x} = \frac{\cos x(\tan x - x)}{\sin^2 x} > 0$, by **(i)**

$\therefore$ f(x) is a strictly increasing function.

$\therefore$ $f(0) < f(x) < f\left(\frac{\pi}{2}\right) \Rightarrow \lim_{x\to 0}\frac{x}{\sin x} < \frac{x}{\sin x} < \frac{\pi/2}{\sin(\pi/2)} \Rightarrow 1 < \frac{x}{\sin x} < \frac{\pi}{2}$

17. Let $f(x) = |x|^3 - 6x^2 + 11|x| - 6$.

$f(-x) = f(x) \Rightarrow f(x)$ is symmetric about y-axis .

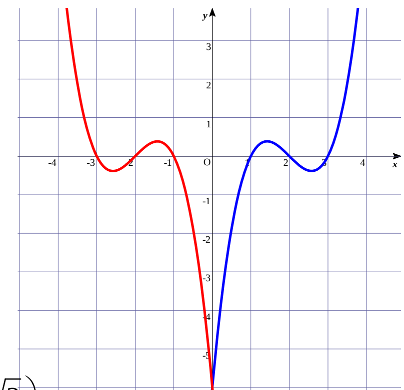
We need to know the case where $x \geq 0$.

When $x \geq 0$, the function becomes

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11$, $\quad f''(x) = 6x - 12$

$f'(x) = 0 \Rightarrow \frac{6 \pm \sqrt{3}}{3}$



$f''\left(\frac{6+\sqrt{3}}{3}\right) = 6\left(\frac{6+\sqrt{3}}{3}\right) - 12 = 2\sqrt{3} > 0$, $\quad f''\left(\frac{6-\sqrt{3}}{3}\right) = 6\left(\frac{6-\sqrt{3}}{3}\right) - 12 = -2\sqrt{3} < 0$

$\therefore \quad f\left(\frac{6-\sqrt{3}}{3}\right) \approx 0.385$ is a local maximum and $f\left(\frac{6+\sqrt{3}}{3}\right) \approx -0.385$ is a local minimum.

As f(x) is symmetric about y-axis,

$\therefore \quad f\left(-\frac{6-\sqrt{3}}{3}\right) \approx 0.385$ is also a local maximum and $f\left(-\frac{6+\sqrt{3}}{3}\right) \approx -0.385$ is also a local minimum.

Since $f(x) \to \pm\infty$ as $x \to +\infty$, there is no absolute maximum.

Since $f(0) = -6 < -0.385$. The absolute minimum is $y = -6$, when $x = 0$.

18. **(a)** $f(x) = x^3 - 6x + 2$, $f'(x) = 3x^2 - 6$, $f''(x) = 6x$, $f'''(x) = 6$

$f'(\pm\sqrt{2}) = 0$, $f''(\sqrt{2}) > 0$, $f''(-\sqrt{2}) < 0$, $f(\sqrt{2}) = 2 - 4\sqrt{2}$, $f(-\sqrt{2}) = 2 + 4\sqrt{2}$

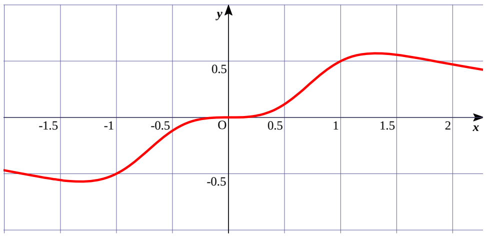
Local max. point = $(-\sqrt{2},\ 2+4\sqrt{2})$. Local min. point = $(\sqrt{2},\ 2-4\sqrt{2})$

As $x \to \pm\infty$, $f(x) \to \pm\infty$, there is no absolute max. or minimum.

$f''(0) = 0$, $f'''(0) = 6 > 0$, $f(0) = 2$.

Point of inflection is (0, 2) .

(b) $f(x) = \frac{x^3}{x^4+1}$



The function is symmetric about origin as f(-x) = -f(x) .

$f'(x) = -\frac{x^2(x^4-3)}{(x^4+1)^2}$, $f''(x) = \frac{2x(x^8-12x^4+3)}{(x^4+1)^3}$

$f'(x) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt[4]{3}$

$f''(\sqrt[4]{3}) = -\frac{3\sqrt[4]{3}}{4} < 0$, $f''(-\sqrt[4]{3}) = \frac{3\sqrt[4]{3}}{4} > 0$, $f(\sqrt[4]{3}) = \frac{\sqrt[4]{27}}{4}$, $f(-\sqrt[4]{3}) = -\frac{\sqrt[4]{27}}{4}$

As $x \to \pm\infty$, $f(x) \to 0$. There is no sign change of $f'(x)$ around $x = 0$. (not max/min)

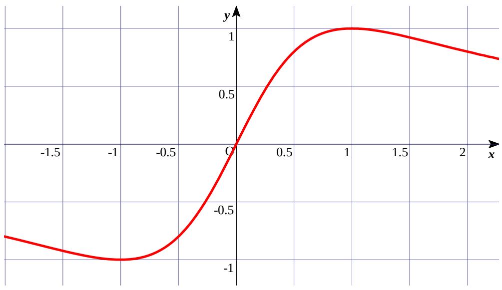
Absolute min. point = $\left(-\sqrt[4]{3}, -\frac{\sqrt[4]{27}}{4}\right)$. Absolute max. point = $\left(\sqrt[4]{3}, \frac{\sqrt[4]{27}}{4}\right)$.

$f'(x) = 0 \Rightarrow x = 0$ or $x^8 - 12x + 3 = 0 \Rightarrow x = 0$ or $x^4 = 6 \pm \sqrt{33}$

Real roots are : $x = 0$, $x = \pm\sqrt[4]{6\pm\sqrt{33}}$ and there are sign changes of $f''(x)$ around these points.

There are points of inflection at $x = 0$, $x = \pm\sqrt[4]{6\pm\sqrt{33}}$.

(c) $f(x) = \frac{2x}{1+x^2}$



$f'(x) = -\frac{2(x+1)(x-1)}{(1+x^2)^2}$, $f''(x) = \frac{4x(x^2-3)}{(1+x^2)^3}$

$f'(x) = 0 \Rightarrow x = \pm 1$.

$f''(\pm 1) = \mp 1$, $f(\pm 1) = \mp 1$

As $x \to \pm\infty$, $f(x) \to 0$.

Absolute max. point = (-1, -1) . Absolute min. point = (1, 1)

$f''(x) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{3}$ and there are sign changes of $f''(x)$ around these points.

$f(0) = 0$ and $f\left(\pm\sqrt{3}\right) = \pm\frac{1}{2}\sqrt{3}$.

The points of inflections are $(0, 0)$, $\left(\pm\sqrt{3},\ \pm\frac{1}{2}\sqrt{3}\right)$.

19. Let $y = \sin mx \csc x$ ($m \in \mathbb{Z}$)

$\therefore$ $y' = \sin mx\,(-\csc x \cot x) + \csc x\,(m \cos mx)$

$\therefore$ $y' = 0 \Rightarrow \csc x\,(m \cos mx - \cot x \sin mx) = 0 \Rightarrow \csc x = 0$ or $m \cos mx - \cot x \sin mx = 0$

$\Rightarrow \csc x = 0$ or $\tan mx = m \tan x$

$y'' = \csc x\,[-m^2 \sin mx - \cot x\,(m \cot mx) + \sin mx \csc^2 x\,] + (m \cos mx - \cot x \sin mx)\,(-\csc x \cot x)$

When $\csc x = 0$, $\sin x \to \infty$, which has no solution.

When $\tan mx = m \tan x$, there are infinite number of solutions for x .

$y'' > 0$ or $y'' < 0$ according to the value of x taken , which yields the minima and maxima of y .

Now, $\tan mx = m \tan x \Rightarrow \tan^2 mx = m^2 \tan^2 x$

$$\Rightarrow \frac{\sin^2 mx}{\sin^2 x} = m^2\frac{\cos^2 mx}{\cos^2 x} = m^2\frac{1+\tan^2 x}{1+\tan^2 mx} = m^2\frac{1+\tan^2 x}{1+m^2\tan^2 x} \quad \ldots. \quad (1)$$

Since $m \neq 0$, $m^2 \geq 1$, $m^2 \tan^2 x \geq \tan^2 x$, $1 + m^2 \tan^2 x \geq 1 + \tan^2 x$

For (1), $\frac{\sin^2 mx}{\sin^2 x} = m^2\frac{1+\tan^2 x}{1+m^2\tan^2 x} \leq m^2 \Rightarrow \sin^2 mx \leq m^2 \sin^2 x$. Equality also holds for $m = 0$.

20. The function $x^m y^n$, where $x, y > 0$ and $x + y = k$ $(m, n, k > 0)$ is equivalent to $f(x) = x^m (k - x)^n$.

$f'(x) = m(k-x)^n x^{m-1} - n(k-x)^{n-1}x^m = x^{m-1}(k-x)^{n-1}[m(k-x) - nx] = x^{m-1}(k-x)^{n-1}[mk - (m+n)x] = 0$

$\Rightarrow x = 0$ or $x = k$ or $x = \frac{mk}{m+n}$

Since $x > 0$, $k - x = y > 0 \Rightarrow x < k$. Therefore $x = \frac{mk}{m+n}$

When $x < \frac{mk}{m+n}$, $mk - (m+n)x > 0$. Therefore $f'(x) > 0$. Note that $x^{m-1}(k-x)^{n-1} > 0$ in $f'(x)$.

When $x < \frac{mk}{m+n}$, $mk - (m+n)x > 0$. Therefore $f'(x) < 0$.

$\therefore$ $f(x)$ is a max. when $x = \frac{mk}{m+n}$. Max of $f(x) = f\left(\frac{mk}{m+n}\right) = \left(\frac{mk}{m+n}\right)^m\left(k - \frac{mk}{m+n}\right)^n = \frac{m^m n^n k^{m+n}}{(m+n)^{m+n}}$.

21. $S = \sum_{k=1}^{n}(x - a_k)^2 \Rightarrow \frac{dS}{dx} = \sum_{k=1}^{n} 2(x - a_k) = 2\left(nx - \sum_{k=1}^{n} a_k\right)$, $\frac{d^2S}{dx^2} = 2n$

$\frac{dS}{dx} = 0 \Rightarrow$ $x = x_0 = \frac{\sum_{k=1}^{n} a_k}{n}$, $\left.\frac{d^2S}{dx^2}\right|_{x=x_0} = 2n$. $\therefore$ S is minimum at $x = x_0$.

22. $y = Ax^{1/2} + Bx^{-1/2} \Rightarrow y' = \frac{A}{2x^{1/2}} - \frac{B}{2x^{3/2}} \Rightarrow y'' = -\frac{A}{4x^{3/2}} + \frac{B}{4x^{5/2}} = -\frac{Ax - 3B}{4x^{5/2}}$

When $x = 4, y = 6$ $\Rightarrow$ $6 = A\,4^{1/2} + B\,4^{-1/2}$ $\Rightarrow$ $4A + B = 12$ …. (1)

For inflectional point at $x = 4$, $y'' = 0$, $4A - 3B = 0$ …. (2)

Solving, we get $A = 9/4$, $B = 3$.

23. $(1 + x^2)\, y = 1 - x \quad \Rightarrow y = \frac{1-x}{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{x^2 - 2x - 1}{(1+x^2)^2}$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2(x+1)(x^2-4x+1)}{(1+x^2)^3} = -\frac{2(x+1)\left[x-(2-\sqrt{3})\right]\left[x-(2+\sqrt{3})\right]}{(1+x^2)^3}$$

$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow x = -1,\ 2-\sqrt{3},\ 2+\sqrt{3}$ and these are points of inflection since there are sign changes as x passes through these points .

These points of inflections : $(-1,1)$, $\left(2-\sqrt{3}, \frac{1+\sqrt{3}}{4}\right)$, $\left(2+\sqrt{3}, \frac{1-\sqrt{3}}{4}\right)$ lies on the same straight line as the gradient of any of two points is $-\frac{1}{4}$.

24. (1) $f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)}{h} = 1$.

(2) $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left[\frac{f(x)+f(h)}{1-f(x)f(h)} - f(x)\right] = \lim_{h\to 0}\frac{1}{h}\left[\frac{f(x)+f(h)-f(x)+f^2(x)f(h)}{1-f(x)f(h)}\right]$

$= \lim_{h\to 0}\frac{f(h)}{h}\left[\frac{1+f^2(x)}{1-f(x)f(h)}\right] == \lim_{h\to 0}\frac{f(h)}{h}\lim_{h\to 0}\left[\frac{1+f^2(x)}{1-f(x)f(h)}\right] = 1\times\frac{1+f^2(x)}{1-f(x)f(0)}$, by (1).

$= 1 + f^2(x)$.

(3) From (2), $\frac{dy}{dx} = 1 + y^2 \quad \Rightarrow \frac{dy}{1+y^2} = dx \quad \Rightarrow \int\frac{dy}{1+y^2} = \int dx \quad \Rightarrow x = \tan^{-1} y + c$

$\Rightarrow \quad x = \tan^{-1}[f(x)] + c$

Put $x = 0$, $0 = \tan^{-1}[f(0)] + c \Rightarrow c = 0$. $\therefore$ $f(x) = \tan x$.

(4) $f(x) = \tan x$ is obviously convex in $x \in \left(0, \frac{\pi}{2}\right)$.

25. **(a)** **(i)** $f(x) = f(x_1) \Leftrightarrow f_1(x) \ge f_2(x) \Leftrightarrow \sqrt{(x-a)^2 + b^2} \ge \sqrt{(x-c)^2 + d^2}$

$\Leftrightarrow (x-a)^2 + b^2 \ge (x-c)^2 + d^2 \Leftrightarrow x \le \frac{a^2+b^2-c^2-d^2}{2(a-c)} = x_0$,where $a > c$.

$f(x) = f(x_2) \Leftrightarrow f_1(x) \le f_2(x) \Leftrightarrow x \ge \frac{a^2+b^2-c^2-d^2}{2(a-c)} = x_0$,where $a > c$.

(ii) Given that $c < a$.

(1) If $c < a < x_0$, then $x_1 = a$

(2) If $c < x_0 < a$, then $x_1 = x_0$

(3) If $x_0 < c < a$, then $x_1 = c$

(b) Let the line L be the x-axis. Let $P_1(a, b)$ and $P_2(c, d)$. Let $P(x, 0)$ be a point on L.

$$P_1P = f_1(x) = \sqrt{(x-a)^2 + b^2}, \qquad P_2P = f_2(x) = \sqrt{(x-c)^2 + d^2}$$

The problem is reduced to **(a)** in finding the minimum of $f(x) = \text{Max}\{f_1(x), f_2(x)\}$.

26. **(a)** $\displaystyle\sum_{r=1}^{n} 2\sin\frac{1}{2}x\cos rx = \sum_{r=1}^{n}\left[\sin\left(r+\frac{1}{2}\right)x - \sin\left(r-\frac{1}{2}\right)x\right] = \sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x$

(b) **(i)** $\displaystyle S_n'(x) = \sum_{r=1}^{n}\left(\frac{\sin rx}{r}\right)' = \sum_{r=1}^{n}\cos rx = \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right] - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$

$$S_n'(x0) = 0 \Rightarrow \sin\left[\left(n+\frac{1}{2}\right)x_0\right] - \sin\frac{1}{2}x_0 = 0 \Rightarrow \sin nx_0\cos\frac{1}{2}x_0 + \cos nx_0\sin\frac{1}{2}x_0 - \sin\frac{1}{2}x_0 = 0$$

$\displaystyle \Rightarrow \sin nx_0 = \frac{\sin\frac{1}{2}x_0(1-\cos nx_0)}{\cos\frac{1}{2}x_0} > 0$ as $x_0 \in (0, \pi)$.

(ii) Since $S_{m+1}(0) = 0$ and x_0 is the absolute minimum point,

$S_{m+1}(x_0) \le 0$. that is, $\displaystyle S_m(x_0) + \frac{\sin(m+1)x_0}{m+1} \le 0$

But $\sin(m+1)x_0 \ge 0$. Therefore $S_m(x_0) \le 0$.

Hence if $S_m(x) > 0$ for all $x \in (0, \pi)$, $S_{m+1}(x_0)$ would not be the absolute minimum for any $x_0 \in (0, \pi)$. and $S_{m+1}(x)$ would be the absolute minimum only at $x = 0$ or $x = \pi$.

(Note that $S_{m+1}(0) = S_{m+1}(\pi) = 0$.)

(iii) Suppose $S_k(x) > 0$ for all $x \in (0, \pi)$.

By (ii), $S_{k+1}(x)$ attains its absolute minimum at $x = 0$ and $x = \pi$.

So for all $x \in (0, \pi)$, $S_{k+1}(x) > S_k(x) = 0$.

Hence the statement is also true for $n = k + 1$.

28. $\displaystyle I = \frac{A}{r^2} + \frac{8A}{(6-r)^2}$, where $A > 0$, I is the intensity of illumination, r is the distance from the first light.

In order that I is well-defined, $0 < r < 6$.

$$\frac{dI}{dr} = -\frac{2A}{r^3} + \frac{16A}{(6-r)^3} = \frac{2A\left[(2r)^3 - (6-r)^3\right]}{r^3(6-r)^3} = \frac{2A(3r-6)\left[(2r)^2 + (2r)(6-r) + (6-r)^2\right]}{r^3(6-r)^3} = \frac{18A(r-2)(r^2+12)}{r^3(6-r)^3}$$

$\displaystyle r = 2 \Rightarrow \frac{dI}{dr} = 0,\quad r > 2 \Rightarrow \frac{dI}{dr} > 0$ and $\displaystyle 0 < r < 2 \Rightarrow \frac{dI}{dr} < 0$

$\therefore$ The total illumination, I , is a minimum when the distance is 2 m from the first light.